Lecture 1

Review of Basics: Mathematical Foundations

Wednesday, January 19, 2000

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Readings:
Appendix 1-4, Foley *et al*
Slide Set 1, VanDam

Lecture Outline

- **Student Information**
  - Instructional demographics: background, department, academic interests
  - Requests for special topics
- **In-Class Exercise: Turn to A Partner**
  - Applications of CG to human-computer interaction (HCI) problems
  - Common advantages and obstacles
- **Quick Review: Basic Analytic Geometry and Linear Algebra for CG**
  - Vector spaces and affine spaces
    - Subspaces
    - Linear independence
    - Bases and orthonormality
  - Equations for objects in affine spaces
    - Lines
    - Planes
  - Dot products and distance measures (norms, equations)

Introductions

- **Student Information (Confidential)**
  - Instructional demographics: background, department, academic interests
  - Requests for special topics
    - Lecture
    - Project
- **On Information Form, Please Write**
  - Your name
  - What you wish to learn from this course
  - What experience (if any) you have with
    - Basic computer graphics
    - Linear algebra
  - What experience (if any) you have in using CG (rendering, animation, visualization) packages
  - What programming languages you know *well*
  - Any specific applications or topics you would like to see covered

In-Class Exercise

- **Turn to A Partner**
  - 2-minute exercise
  - Briefly introduce yourselves (2 minutes)
  - 3-minute discussion
  - 10-minute go-round
  - 3-minute follow-up
- **Questions**
  - 2 applications of CG systems to HCI problem in your area
  - *Common* advantage and obstacle
- **Project LEA/RN™ Exercise, Iowa State [Johnson and Johnson, 1998]**
  - Formulate an answer *individually*
  - Share your answer with your partner
  - Listen carefully to your partner's answer
  - Create a new answer through discussion
  - Account for your discussion by being prepared to be called upon

Quick Review: Basic Linear Algebra for CG

- **Readings: Appendix A.1 – A.4, Foley *et al***
- **A.1 Vector Spaces and Affine Spaces**
  - Equations of lines, planes
  - Vector subspaces and affine subspaces
- **A.2 Standard Constructions in Vector Spaces**
  - Linear independence and spans
  - Coordinate systems and bases
- **A.3 Dot Products and Distances**
  - Dot product in $\mathbb{R}^n$
  - Norms in $\mathbb{R}^n$
- **A.4 Matrices**
  - Binary matrix operations: basic arithmetic
  - Unary matrix operations: transpose and inverse
- **Application: Transformations and Change of Coordinate Systems**

Vector Spaces and Affine Spaces

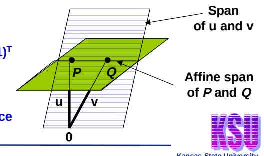
- **Vector Space: Set of Points Admitting Addition, Multiplication by Constant**
  - Components
    - Set V (of vectors **u**, **v**, **w**): addition, scalar multiplication defined on members
    - Vector addition: **v** + **w**
    - Scalar multiplication: α**v**
  - Properties (necessary and sufficient conditions)
    - Addition: associative, commutative, identity (**0** vector such that $\forall$ **v** . **0** + **v** = **v**), admits inverses ($\forall$ **v** . $\exists$**w** . **v** + **w** = **0**)
    - Scalar multiplication: satisfies $\forall$ α, β, **v** . $(\alpha\beta)$**v** = $\alpha(\beta$**v**), $\forall$**v** . 1**v** = **v**, $\forall$ α, β, **v** . $(\alpha + \beta)$**v** = α**v** + β**v**, $\forall$ α, β, **v** . α(**v** + **w**) = α**v** + α**w**
  - Linear combination: $\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n$
- **Affine Space: Set of Points Admitting Geometric Operations (No "Origin")**
  - Components
    - Set V (of points P, Q, R) *and* associated vector space
    - Operators: vector difference, point-vector addition
  - Affine combination (of P and Q by $t \in \mathbb{R}$): $P + t(Q - P)$
  - *NB*: any vector space (V, +, ·) can be made into affine space (points(V), V)

Linear and Planar Equations in Affine Spaces

- Equation of Line in Affine Space
  - Let P, Q be points in affine space
  - Parametric form (real-valued parameter t)
    - Set of points of form $(1 - t)P + tQ$
    - Forms line passing through P and Q
  - Example
    - Cartesian plane of points (x, y) is an affine space
    - Parametric line between (a, b) and (c, d): $L = \{((1 - t)a + tc, (1 - t)b + td) \mid t \in \mathbb{R}\}$
- Equation of Plane in Affine Space
  - Let P, Q, R be points in affine space
  - Parametric form (real-valued parameters s, t)
    - Set of points of form $(1 - s)((1 - t)P + tQ) + sR$
    - Forms plane containing P, Q, R

Vector Space Spans and Affine Spans

- Vector Space Span
  - Definition – set of all linear combinations of a set of vectors
  - Example: vectors in $\mathbb{R}^3$
    - Span of single (nonzero) vector **v**: line through the origin containing **v**
    - Span of pair of (nonzero, noncollinear) vectors: plane through the origin containing both
    - Span of 3 of vectors in general position: all of $\mathbb{R}^3$
- Affine Span
  - Definition – set of all affine combinations of a set of points $P_1, P_2, \ldots, P_n$ in an affine space
  - Example: vectors, points in $\mathbb{R}^3$
    - Standard affine plan of points $(x, y, 1)^T$
    - Consider points P, Q
    - Affine span: line containing P, Q
    - Also intersection of span, affine space



Independence

- Linear Independence
  - Definition: (linearly) dependent vectors
    - Set of vectors $\{v_1, v_2, \ldots, v_n\}$ such that one lies in the span of the rest
    - $\exists\, v_i \in \{v_1, v_2, \ldots, v_n\} .\ v_i \in \text{Span}(\{v_1, v_2, \ldots, v_n\} \sim \{v_i\})$
  - (Linearly) independent: $\{v_1, v_2, \ldots, v_n\}$ not dependent
- Affine Independence
  - Definition: (affinely) dependent points
    - Set of points $\{v_1, v_2, \ldots, v_n\}$ such that one lies in the (affine) span of the rest
    - $\exists\, P_i \in \{P_1, P_2, \ldots, P_n\} .\ P_i \in \text{Span}(\{P_1, P_2, \ldots, P_n\} \sim \{P_i\})$
  - (Affinely) independent: $\{P_1, P_2, \ldots, P_n\}$ not dependent
- Consequences of Linear Independence
  - Equivalent condition: $\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n = 0 \Leftrightarrow \alpha_1 = \alpha_2 = \ldots = \alpha_n = 0$
  - Dimension of span is equal to the number of vectors

Subspaces

- Intuitive Idea
  - $\mathbb{R}^n$: vector or affine space of "equal or lower dimension"
  - Closed under constructive operator for space
- Linear Subspace
  - Definition
    - Subset S of vector space $(V, +, \cdot)$
    - Closed under addition (+) and scalar multiplication ($\cdot$)
  - Examples
    - Subspaces of $\mathbb{R}^3$: origin (0, 0, 0), line through the origin, plane containing origin, $\mathbb{R}^3$ itself
    - For vector **v**, $\{\alpha v \mid \alpha \in \mathbb{R}\}$ is a subspace (why?)
- Affine Subspace
  - Definition
    - Nonempty subset S of vector space $(V, +, \cdot)$
    - Closure S' of S under point subtraction is a linear subspace of V
  - *Important affine subspace of* $\mathbb{R}^4$ (foundation of Chapter 5): $\{(x, y, z, 1)\}$

Bases

- Spanning Set (of Set S of Vectors)
  - Definition: set of vectors for which any vector in Span(S) can be expressed as linear combination of vectors in spanning set
  - Intuitive idea: spanning set "covers" Span(S)
- Basis (of Set S of Vectors)
  - Definition
    - *Minimal* spanning set of S
    - Minimal: any smaller set of vectors has smaller span
  - Alternative definition: linearly independent spanning set
- Exercise
  - Claim: basis of subspace of vector space is always linearly independent
  - Proof: by contradiction (suppose basis is dependent… not minimal)
- Standard Basis for $\mathbb{R}^3$
  - $E = \{e_1, e_2, e_3\}$, $e_1 = (1, 0, 0)^T$, $e_2 = (0, 1, 0)^T$, $e_3 = (0, 0, 1)^T$
  - *How to use this as coordinate system?*

Coordinates and Coordinate Systems

- Coordinates Using Bases
  - Coordinates
    - Consider basis $B = \{v_1, v_2, \ldots, v_n\}$ for vector space
    - Any vector **v** in the vector space can be expressed as linear combination of vectors in **B**
    - Definition: coefficients of linear combination are coordinates
  - Example
    - $E = \{e_1, e_2, e_3\}$, $e_1 = (1, 0, 0)^T$, $e_2 = (0, 1, 0)^T$, $e_3 = (0, 0, 1)^T$
    - Coordinates of (a, b, c) with respect to **E**: $(a, b, c)^T$
- Coordinate System
  - Definition: set of independent points in affine space
  - Affine span of coordinate system is entire affine space
- Exercise
  - Derive basis for associated vector space of arbitrary coordinate system
  - (Hint: consider definition of affine span…)

Dot Products and Distances

- Dot Product in $\mathbb{R}^n$
  - Given: vectors $\mathbf{u} = (u_1, u_2, \ldots, u_n)^T$, $\mathbf{v} = (v_1, v_2, \ldots, v_n)^T$
  - Definition
    - Dot product $\mathbf{u} \bullet \mathbf{v} \equiv u_1v_1 + u_2v_2 + \ldots + u_nv_n$
    - Also known as inner product
    - In $\mathbb{R}^n$, called scalar product
- Applications of the Dot Product
  - Normalization of vectors
  - Distances
  - Generating equations
  - See Appendix A.3, Foley *et al* (FVD)

Norms and Distance Formulas

- Length
  - Definition
    - $\|v\| = \sqrt{v \bullet v}$
    - $\mathbf{v} \bullet \mathbf{v} = \sum_i v_i^2$
  - *aka* Euclidean norm
- Applications of the Dot Product
  - Normalization of vectors: division by scalar length $\| \mathbf{v} \|$ converts to unit vector
  - Distances
    - Between points: $\| Q - P \|$
    - From points to planes
  - Generating equations (e.g., point loci): circles, hollow cylinders, etc.
  - Ray / object intersection equations
  - See A.3.5, FVD

Intersection Formulas

- Intersection Equations: Rays and Objects
  - Application
    - Simple ray tracing (*aka* ray casting)
    - Chapters 15-16, FVD
  - General case: substitute point $P + t\mathbf{v}$ to derive parametric form from implicit equation of object $F(x, y, z, 1) = 0$
- Example
  - Circle
    - Implicit equation of circle: $(X - P) \bullet (X - P) = r^2$
    - Parametric equation of line: $S(t) = Q + t\,\mathbf{v}$
    - Plug line LHS into circle equation: $(S(t) - P) \bullet (S(t) - P) = r^2$
    - Solve for t using quadratic formula
  - Sphere, polygon: 15.10.1 FVD

Orthonormal Bases

- Orthogonality
  - Given: vectors $\mathbf{u} = (u_1, u_2, \ldots, u_n)^T$, $\mathbf{v} = (v_1, v_2, \ldots, v_n)^T$
  - Definition
    - $\mathbf{u}$, $\mathbf{v}$ are orthogonal if $\mathbf{u} \bullet \mathbf{v} = 0$
    - In $\mathbb{R}^2$, angle between orthogonal vectors is 90°
- Orthonormal Bases
  - Necessary and sufficient conditions
    - $B = \{b_1, b_2, \ldots, b_n\}$ is basis for given vector space
    - Every pair (b_i, b_j) is orthogonal
    - Every vector b_i is of unit magnitude ($\| v_i \| = 1$)
  - Convenient property: can just take dot product $\mathbf{v} \bullet b_i$ to find coefficients in linear combination (coordinates with respect to B) for vector $\mathbf{v}$

Terminology

- Human Computer [Intelligent] Interaction (HCI, HCII)
- Some Basic Analytic Geometry and Linear Algebra for CG
  - Vector space (VS) – collection of vectors admitting addition, scalar multiplication and observing VS axioms
  - Affine space (AS) – collection of points with associated vector space admitting vector difference, point-vector addition and observing AS axioms
  - Linear subspace – nonempty subset S of VS $(V, +, \cdot)$ closed under + and ·
  - Affine subspace – nonempty subset S of VS $(V, +, \cdot)$ such that closure S' of S under point subtraction is a linear subspace of V
  - Span – set of all linear combinations of set of vectors
  - Linear independence – property of set of vectors that none lies in span of others
  - Basis – minimal spanning set of set of vectors
  - Dot product – scalar-valued inner product $\langle u, v \rangle \equiv \mathbf{u} \bullet \mathbf{v} \equiv u_1v_1 + u_2v_2 + \ldots + u_nv_n$
  - Orthogonality – property of vectors $\mathbf{u}$, $\mathbf{v}$ that $\mathbf{u} \bullet \mathbf{v} = 0$
  - Orthonormality – basis containing pairwise-orthogonal unit vectors
  - Length (Euclidean norm) – $\|v\| = \sqrt{v \bullet v}$

Summary Points

- Student Information
- In-Class Exercise: Turn to A Partner
  - Applications of CG to 2 human-computer interaction (HCI) problems
  - Common advantages
  - After-class exercise: think about common obstacles (send e-mail or post)
- Quick Review: Some Basic Analytic Geometry and Linear Algebra for CG
  - Vector spaces and affine spaces
    - Subspaces
    - Linear independence
    - Bases and orthonormality
  - Equations for objects in affine spaces
    - Lines
    - Planes
  - Dot products and distance measures (norms, equations)
- Next Lecture: Geometry, Scan Conversion (Lines, Polygons)