# Lecture 17 of 42

## SVM Continued and Intro to Bayesian Learning: *Max a Posteriori* and Max Likelihood Estimation

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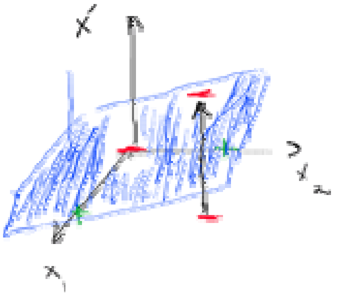
Readings:
Sections 6.1-6.5, Mitchell

# Lecture Outline

- **Read Sections 6.1-6.5, Mitchell**
- **Overview of Bayesian Learning**
  - **Framework: using probabilistic criteria to generate hypotheses of all kinds**
  - **Probability: foundations**
- **Bayes's Theorem**
  - **Definition of conditional (posterior) probability**
  - **Ramifications of Bayes's Theorem**
    - **Answering probabilistic queries**
    - **MAP hypotheses**
- **Generating Maximum *A Posteriori* (MAP) Hypotheses**
- **Generating Maximum Likelihood Hypotheses**
- **Next Week: Sections 6.6-6.13, Mitchell; Roth; Pearl and Verma**
  - **More Bayesian learning: MDL, BOC, Gibbs, Simple (Naïve) Bayes**
  - **Learning over text**

# Review: Support Vector Machines (SVM)

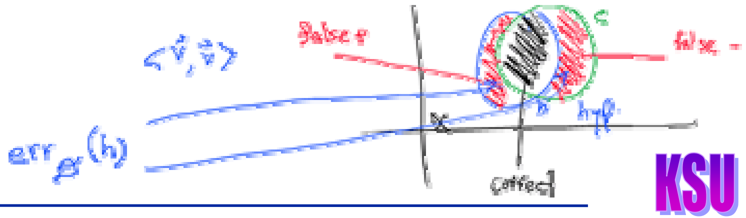


| $m$ | $= \lvert D \rvert$ |
|---|---|
| size(c) | coding ln of conc. |
| $n$ | # attributes |
| $\delta$ | conf bound |
| $\epsilon$ | err bound |

$$\|\vec{v}\|_2 = \sqrt{\vec{v}\cdot\vec{v}} = \sqrt{\sum_{i=1}^{n} v_i^2}$$

true conc.

$$\langle \vec{v}, \vec{v} \rangle$$



# Roadmap

# Selection and Building Blocks

- Restricted Case: Selection Only
  - $\bar{f}(t)$ ≡ average fitness of population at time *t*
  - $m(s, t)$ ≡ number of instances of schema *s* in population at time *t*
  - $\hat{u}(s, t)$ ≡ average fitness of instances of schema *s* at time *t*
- Quantities of Interest
  - Probability of selecting *h* in one selection step

$$P(h) = \frac{f(h)}{\sum_{i=1}^{n} f(h_i)}$$

  - Probability of selecting an instance of *s* in one selection step

$$P(h \in s) = \sum_{h \in (s \cap p_t)} \frac{f(h)}{n \cdot \bar{f}(t)} = \frac{\hat{u}(s, t)}{n \cdot \bar{f}(t)} \cdot m(s, t)$$

  - Expected number of instances of *s* after *n* selections

$$E[m(s, t+1)] = \boxed{\frac{\hat{u}(s, t)}{\bar{f}(t)} \cdot m(s, t)}$$

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# Bayesian Learning

- Framework: Interpretations of Probability [Cheeseman, 1985]
  - Bayesian <u>subjectivist</u> view
    - A <u>measure of an agent's belief</u> in a proposition
    - Proposition denoted by random variable (sample space: range)
    - e.g., *Pr*(*Outlook* = *Sunny*) = 0.8
  - <u>Frequentist</u> view: probability is the ***frequency of observations*** of an event
  - <u>Logicist</u> view: probability is <u>inferential evidence</u> in favor of a proposition
- Typical Applications
  - HCI: learning natural language; intelligent displays; decision support
  - Approaches: prediction; sensor and data fusion (e.g., bioinformatics)
- Prediction: Examples
  - Measure ***relevant parameters***: temperature, barometric pressure, wind speed
  - Make statement of the form *Pr*(*Tomorrow's-Weather* = *Rain*) = 0.5
  - College admissions: *Pr*(*Acceptance*) ≡ *p*
    - <u>Plain beliefs</u>: unconditional acceptance (*p* = 1) or <u>categorical</u> rejection (*p* = 0)
    - Conditional beliefs: depends on reviewer (use probabilistic model)

# Two Roles for Bayesian Methods

- Practical Learning Algorithms
  - Naïve Bayes (*aka* simple Bayes)
  - Bayesian belief network (BBN) structure learning and parameter estimation
  - Combining prior knowledge (prior probabilities) with observed data
    - A way to incorporate background knowledge (BK), *aka* domain knowledge
    - Requires prior probabilities (e.g., annotated rules)
- Useful Conceptual Framework
  - Provides “gold standard” for evaluating other learning algorithms
    - Bayes Optimal Classifier (BOC)
    - Stochastic Bayesian learning: Markov chain Monte Carlo (MCMC)
  - Additional insight into Occam’s Razor (MDL)

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# Probabilistic Concepts versus Probabilistic Learning

- Two Distinct Notions: Probabilistic Concepts, Probabilistic Learning
- Probabilistic Concepts
  - Learned concept is a *function*, $c: X \rightarrow [0, 1]$
  - $c(x)$, the target value, denotes the probability that the label 1 (i.e., *True*) is assigned to $x$
  - Previous learning theory is applicable (with some extensions)
- Probabilistic (i.e., Bayesian) Learning
  - Use of a probabilistic criterion in selecting a hypothesis $h$
    - e.g., “most likely” $h$ given observed data $D$: MAP hypothesis
    - e.g., $h$ for which $D$ is “most likely”: max likelihood (ML) hypothesis
    - May or may not be stochastic (i.e., search process might still be deterministic)
  - NB: $h$ can be deterministic (e.g., a Boolean function) or probabilistic

# Probability:
# Basic Definitions and Axioms

- **Sample Space ($\Omega$): Range of a Random Variable *X***
- **Probability Measure *Pr*(•)**
  - $\Omega$ denotes a range of "events"; *X*: $\Omega$
  - Probability *Pr*, or *P*, is a *measure* over $\Omega$
  - In a general sense, $Pr(X = x \in \Omega)$ is a measure of belief in $X = x$
    - $P(X = x) = 0$ or $P(X = x) = 1$: plain (*aka* categorical) beliefs (can't be revised)
    - All other beliefs are subject to revision
- **Kolmogorov Axioms**
  - 1. $\forall x \in \Omega \,.\, 0 \leq P(X = x) \leq 1$
  - 2. $P(\Omega) \equiv \sum_{x \in \Omega} P(X = x) = 1$
  - 3. $\forall X_1, X_2, \ldots \ni i \neq j \Rightarrow X_i \wedge X_j = \emptyset\,.$

$$P\left(\bigcup_{i=1}^{\infty} X_i\right) = \sum_{i=1}^{\infty} P(X_i)$$

- **Joint Probability: $P(X_1 \wedge X_2) \equiv$ Probability of the Joint Event $X_1 \wedge X_2$**
- **Independence: $P(X_1 \wedge X_2) = P(X_1) \bullet P(X_2)$**

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# Bayes's Theorem

- **Theorem**

$$P(h \mid D) = \frac{P(D \mid h)P(h)}{P(D)} = \frac{P(h \wedge D)}{P(D)}$$

- **$P(h) \equiv$ Prior Probability of Hypothesis *h***
  - Measures initial beliefs (BK) before any information is obtained (hence prior)
- **$P(D) \equiv$ Prior Probability of Training Data *D***
  - Measures probability of obtaining sample *D* (i.e., expresses $\mathcal{D}$)
- **$P(h \mid D) \equiv$ Probability of *h* Given *D***
  - | denotes conditioning - hence P(h | D) is a conditional (*aka* posterior) probability
- **$P(D \mid h) \equiv$ Probability of *D* Given *h***
  - Measures probability of observing *D* given that *h* is correct ("generative" model)
- **$P(h \wedge D) \equiv$ Joint Probability of *h* and *D***
  - Measures probability of observing *D* and of *h* being correct

# Choosing Hypotheses

- **Bayes's Theorem**

$$P(h \mid D) = \frac{P(D \mid h)P(h)}{P(D)} = \frac{P(h \wedge D)}{P(D)}$$

- **MAP Hypothesis**
  - Generally want most probable hypothesis given the training data
  - Define: $\arg\max_{x \in \Omega} [f(x)]$ ≡ the value of $x$ in the sample space Ω with the highest $f(x)$
  - Maximum *a posteriori* hypothesis, $h_{MAP}$

$$\begin{aligned} h_{MAP} &= \arg\max_{h \in H} P(h \mid D) \\ &= \arg\max_{h \in H} \frac{P(D \mid h)P(h)}{P(D)} \\ &= \arg\max_{h \in H} P(D \mid h)P(h) \end{aligned}$$

- **ML Hypothesis**
  - Assume that $p(h_i) = p(h_j)$ for all pairs $i, j$ (uniform priors, i.e., $P_H$ ~ Uniform)
  - Can further simplify and choose the maximum likelihood hypothesis, $h_{ML}$

$$h_{ML} = \arg\max_{h_i \in H} P(D \mid h_i)$$

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# Bayes's Theorem:
# Query Answering (QA)

- **Answering User Queries**
  - Suppose we want to perform intelligent inferences over a database *DB*
    - Scenario 1: *DB* contains records (instances), some "labeled" with answers
    - Scenario 2: *DB* contains probabilities (annotations) over propositions
  - QA: an application of probabilistic inference
- **QA Using Prior and Conditional Probabilities: Example**
  - Query: *Does patient have cancer or not?*
  - Suppose: patient takes a lab test and result comes back positive
    - Correct + result in only 98% of the cases in which disease is actually present
    - Correct - result in only 97% of the cases in which disease is not present
    - Only 0.008 *of the entire population* has this cancer
  - $\alpha \equiv P(\text{false negative for } H_0 \equiv Cancer) = 0.02$ (*NB*: for 1-point sample)
  - $\beta \equiv P(\text{false positive for } H_0 \equiv Cancer) = 0.03$ (*NB*: for 1-point sample)

$$\begin{array}{lll} P(Cancer) = 0.008 & P(+ \mid Cancer) = 0.98 & P(+ \mid \neg Cancer) = 0.03 \\ P(\neg Cancer) = 0.992 & P(- \mid Cancer) = 0.02 & P(- \mid \neg Cancer) = 0.97 \end{array}$$

  - $P(+ \mid H_0)\, P(H_0) = 0.0078$, $P(+ \mid H_A)\, P(H_A) = 0.0298 \Rightarrow h_{MAP} = H_A \equiv \neg Cancer$

# Basic Formulas for Probabilities

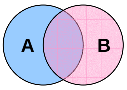
- Product Rule (Alternative Statement of Bayes's Theorem)

$$P(A \mid B) = \frac{P(A \wedge B)}{P(B)}$$

  - Proof: requires axiomatic set theory, as does Bayes's Theorem
- Sum Rule

$$P(A \vee B) = P(A) + P(B) - P(A \wedge B)$$

  - Sketch of proof (immediate from axiomatic set theory)
    - Draw a Venn diagram of two sets denoting events $A$ and $B$
    - Let $A \cup B$ denote the event corresponding to $A \vee B$…



- Theorem of Total Probability
  - Suppose events $A_1, A_2, \ldots, A_n$ are mutually exclusive and exhaustive
    - Mutually exclusive: $i \neq j \Rightarrow A_i \wedge A_j = \varnothing$
    - Exhaustive: $\sum P(A_i) = 1$
  - Then $P(B) = \sum_{i=1}^{n} P(B \mid A_i) \cdot P(A_i)$
  - Proof: follows from product rule and 3rd Kolmogorov axiom

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# MAP and ML Hypotheses: A Pattern Recognition Framework

- Pattern Recognition Framework
  - Automated speech recognition (ASR), automated image recognition
  - Diagnosis
- Forward Problem: One Step in ML Estimation
  - Given: model *h*, observations (data) *D*
  - Estimate: $P(D \mid h)$, the "probability that the model generated the data"
- Backward Problem: Pattern Recognition / Prediction Step
  - Given: model *h*, observations *D*
  - Maximize: $P(h(X) = x \mid h, D)$ for a new $X$ (i.e., find best $x$)
- Forward-Backward (Learning) Problem
  - Given: model space *H*, data *D*
  - Find: $h \in H$ such that $P(h \mid D)$ is maximized (i.e., MAP hypothesis)
- More Info
  - http://www.cs.brown.edu/research/ai/dynamics/tutorial/Documents/HiddenMarkovModels.html
  - Emphasis on a particular *H* (the space of hidden Markov models)

# Bayesian Learning Example: Unbiased Coin [1]

- **Coin Flip**
  - Sample space: $\Omega = \{Head, Tail\}$
  - Scenario: given coin is either fair or has a 60% bias in favor of *Head*
    - $h_1 \equiv$ fair coin: $P(Head) = 0.5$
    - $h_2 \equiv$ 60% bias towards *Head*: $P(Head) = 0.6$
  - Objective: to decide between default (null) and alternative hypotheses
- ***A Priori*** **(*aka* Prior) Distribution on *H***
  - $P(h_1) = 0.75$, $P(h_2) = 0.25$
  - Reflects learning agent's *prior beliefs* regarding *H*
  - Learning is revision of agent's beliefs
- **Collection of Evidence**
  - First piece of evidence: $d \equiv$ a single coin toss, comes up *Head*
  - Q: What does the agent believe now?
  - A: Compute $P(d) = P(d \mid h_1)\, P(h_1) + P(d \mid h_2)\, P(h_2)$

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# Bayesian Learning Example: Unbiased Coin [2]

- **Bayesian Inference: Compute $P(d) = P(d \mid h_1)\, P(h_1) + P(d \mid h_2)\, P(h_2)$**
  - $P(Head) = 0.5 \bullet 0.75 + 0.6 \bullet 0.25 = 0.375 + 0.15 = 0.525$
  - This is the probability of the observation $d = Head$
- **Bayesian Learning**
  - Now apply Bayes's Theorem
    - $P(h_1 \mid d) = P(d \mid h_1)\, P(h_1) \,/\, P(d) = 0.375 / 0.525 = 0.714$
    - $P(h_2 \mid d) = P(d \mid h_2)\, P(h_2) \,/\, P(d) = 0.15 / 0.525 = 0.286$
    - *Belief has been revised downwards for $h_1$, upwards for $h_2$*
    - The agent still thinks that the fair coin is the more likely hypothesis
  - Suppose we were to use the ML approach (i.e., assume equal priors)
    - Belief is revised upwards from 0.5 for $h_1$
    - Data then supports the bias coin better
- **More Evidence: Sequence *D* of 100 coins with 70 heads and 30 tails**
  - $P(D) = (0.5)^{50} \bullet (0.5)^{50} \bullet 0.75 + (0.6)^{70} \bullet (0.4)^{30} \bullet 0.25$
  - Now $P(h_1 \mid d) \ll P(h_2 \mid d)$

# Brute Force MAP Hypothesis Learner

- **Intuitive Idea: Produce Most Likely *h* Given Observed *D***
- **Algorithm Find-MAP-Hypothesis (*D*)**
  - **1. FOR each hypothesis $h \in H$**

    **Calculate the conditional (i.e., posterior) probability:**

$$P(h \mid D) = \frac{P(D \mid h) P(h)}{P(D)}$$

  - **2. RETURN the hypothesis $h_{MAP}$ with the highest conditional probability**

$$h_{MAP} = \arg\max_{h \in H} P(h \mid D)$$

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# Relation to Concept Learning

- **Usual Concept Learning Task**
  - **Instance space *X***
  - **Hypothesis space *H***
  - **Training examples *D***
- **Consider *Find-S* Algorithm**
  - **Given: *D***
  - **Return: most specific h in the version space $VS_{H,D}$**
- **MAP and Concept Learning**
  - **Bayes's Rule: Application of Bayes's Theorem**
  - **What would Bayes's Rule produce as the MAP hypothesis?**
- **Does *Find-S* Output A MAP Hypothesis?**

# Bayesian Concept Learning and Version Spaces

- Assumptions
  - Fixed set of instances $<x_1, x_2, \ldots, x_m>$
  - Let $D$ denote the set of classifications: $D = <c(x_1), c(x_2), \ldots, c(x_m)>$
- Choose $P(D \mid h)$
  - $P(D \mid h) = 1$ if $h$ consistent with $D$ (i.e., $\forall\ x_i \,.\, h(x_i) = c(x_i)$)
  - $P(D \mid h) = 0$ otherwise
- Choose $P(h)$ ~ Uniform
  - Uniform distribution: $P(h) = \frac{1}{|H|}$
  - Uniform priors correspond to “no background knowledge” about $h$
  - Recall: maximum entropy
- MAP Hypothesis

$$P(h \mid D) = \begin{cases} \frac{1}{|VS_{H,D}|} & \text{if } h \text{ is consistent with } D \\ 0 & \text{otherwise} \end{cases}$$

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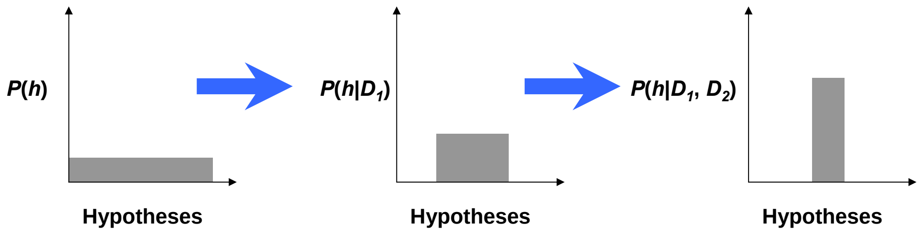
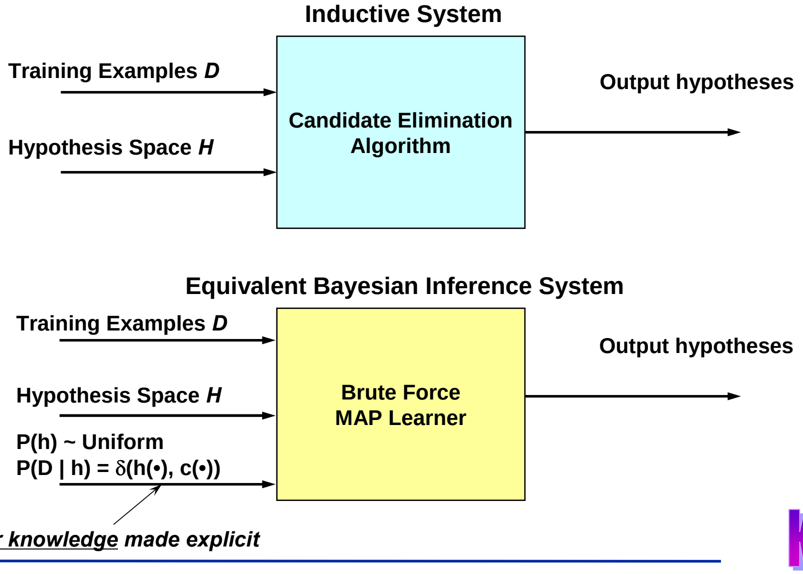
# Evolution of Posterior Probabilities

- Start with Uniform Priors
  - Equal probabilities assigned to each hypothesis
  - Maximum uncertainty (entropy), minimum prior information



- Evidential Inference
  - Introduce data (evidence) $D_1$: belief revision occurs
    - Learning agent revises conditional probability of inconsistent hypotheses to 0
    - Posterior probabilities for remaining $h \in VS_{H,D}$ revised upward
  - Add more data (evidence) $D_2$: further belief revision

# Characterizing Learning Algorithms by Equivalent MAP Learners



**Inductive System**

Training Examples *D*

Hypothesis Space *H*

Candidate Elimination Algorithm

Output hypotheses

**Equivalent Bayesian Inference System**

Training Examples *D*

Hypothesis Space *H*

P(h) ~ Uniform

P(D | h) = δ(h(•), c(•))

Brute Force MAP Learner

Output hypotheses

***<u>Prior knowledge</u> made explicit***

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# Most Probable Classification of New Instances

- **MAP and MLE: Limitations**
  - **Problem so far: "find the most likely hypothesis given the data"**
  - **Sometimes we just want the best classification of a new instance *x*, given *D***
- **A Solution Method**
  - **Find best (MAP) *h*, use it to classify**
  - ***This may not be optimal, though!***
  - **Analogy**
    - **Estimating a distribution using the <u>mode</u> versus the <u>integral</u>**
    - **One finds the maximum, the other the area**
- **Refined Objective**
  - **Want to determine the <u>most probable classification</u>**
  - **Need to *combine* the prediction of all hypotheses**
  - **Predictions must be *weighted by their conditional probabilities***
  - **Result: <u>B</u>ayes <u>O</u>ptimal <u>C</u>lassifier (next time…)**

# Terminology

- Introduction to Bayesian Learning
  - Probability foundations
    - Definitions: subjectivist, frequentist, logicist
    - (3) Kolmogorov axioms
- Bayes's Theorem
  - Prior probability of an event
  - Joint probability of an event
  - Conditional (posterior) probability of an event
- Maximum *A Posteriori* (MAP) and Maximum Likelihood (ML) Hypotheses
  - MAP hypothesis: highest conditional probability given observations (data)
  - ML: highest likelihood of generating the observed data
  - ML estimation (MLE): estimating parameters to find ML hypothesis
- Bayesian Inference: Computing Conditional Probabilities (CPs) in A Model
- Bayesian Learning: Searching Model (Hypothesis) Space using CPs

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# Summary Points

- Introduction to Bayesian Learning
  - Framework: using probabilistic criteria to search *H*
  - Probability foundations
    - Definitions: subjectivist, objectivist; Bayesian, frequentist, logicist
    - Kolmogorov axioms
- Bayes's Theorem
  - Definition of conditional (posterior) probability
  - Product rule
- Maximum *A Posteriori* (MAP) and Maximum Likelihood (ML) Hypotheses
  - Bayes's Rule and MAP
  - Uniform priors: allow use of MLE to generate MAP hypotheses
  - Relation to version spaces, candidate elimination
- Next Week: 6.6-6.10, Mitchell; Chapter 14-15, Russell and Norvig; Roth
  - More Bayesian learning: MDL, BOC, Gibbs, Simple (Naïve) Bayes
  - Learning over text